

# Fluctuation of density of states for 1d Schrödinger operators

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## Abstract

We consider the 1d Schrödinger operator with random decaying potential and compute the 2nd term asymptotics of the density of states, which shows substantial differences between the cases  $\alpha > \frac{1}{2}$ ,  $\alpha < \frac{1}{2}$  and  $\alpha = \frac{1}{2}$ .

## 1 Introduction

1d Schrödinger operators with random decaying potentials have rich spectral structures depending the decay rate of the potential, so that there are many studies on this topic (e.g, [2] and references therein). In this paper we consider the following operator :

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \text{ on } L^2(\mathbf{R})$$

where  $a \in C^\infty(\mathbf{R})$ ,  $a(-t) = a(t)$ ,  $t > 0$ ,  $a$  is non-decreasing for  $t > 0$ , and  $a(t) = t^{-\alpha}(1 + o(1))$ ,  $t \rightarrow \infty$ ,  $\alpha > 0$ .  $F \in C^\infty(M)$  on a torus  $M$  such that

$$\langle F \rangle := \int_M F(x)dx = 0,$$

and  $(X_t)_{t \in \mathbf{R}}$  is a Brownian motion on  $M$  with generator  $L$ . The spectrum of  $H$  on  $[0, \infty)$  is a.c. for  $\alpha > \frac{1}{2}$ , pure point for  $\alpha < \frac{1}{2}$ , and for  $\alpha = \frac{1}{2}$ , pure point on  $[0, E_c]$  and s.c. on  $[E_c, \infty)$  for some  $E_c \geq 0$  [5]. For the level statistics problem, the point process  $\xi_L$ , whose atoms are composed of the rescaled eigenvalues of the finite volume restriction of  $H$ , converges to clock process ( $\alpha > \frac{1}{2}$ ), Sine $_\beta$ -process ( $\alpha = \frac{1}{2}$ ), and Poisson process ( $\alpha < \frac{1}{2}$ ) [3, 6, 4]. Let  $H_n := H|_{[0, n]}$  be the

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restriction of  $H$  on  $L^2[0, n]$  with Dirichlet boundary condition. Pick  $0 < \kappa_1 < \kappa_2$  arbitrary and set

$$N_n(\kappa_1, \kappa_2) := \#\{\text{eigenvalues of } H_n \text{ in } (\kappa_1^2, \kappa_2^2)\}.$$

Since the integrated density of states  $N$  of  $H$  is equal to  $N(E) := \lim_{n \rightarrow \infty} \frac{1}{n} \#\{\text{eigenvalues of } H_n \leq E\} = \pi^{-1} \sqrt{E}$  as far as  $\alpha > 0$ , we have

$$N_n(\kappa_1, \kappa_2) = \frac{n}{\pi} (\kappa_2 - \kappa_1) (1 + o(1)), \quad n \rightarrow \infty. \quad (1.1)$$

The purpose of this paper is to study the 2nd term asymptotics of this equation. This problem is often studied in the context of random matrix theory (e.g., [1]). In what follows, we state the result which is divided into the following three cases :  $\alpha > \frac{1}{2}$ ,  $\alpha = \frac{1}{2}$ , and  $\alpha < \frac{1}{2}$ .

(1) **Super-critical decay** ( $\alpha > \frac{1}{2}$ ) : We need to consider suitable subsequences as we did in the study of level statistics :

#### Assumption A

A subsequence  $\{n_k\}_{k=1}^\infty$  satisfies  $\lim_{k \rightarrow \infty} n_k = \infty$  and as  $k \rightarrow \infty$ ,

$$\{\kappa_j n_k\}_\pi = \gamma_j + o(1),$$

for some  $\gamma_j \in [0, \pi)$ ,  $j = 1, 2$ . Here we set  $\{x\}_\pi := x - \lfloor x \rfloor_\pi \cdot \pi$ ,  $\lfloor x \rfloor_\pi := \lfloor x/\pi \rfloor$ .

We further need to introduce a new quantity. Let  $\theta_t(\kappa)$  be the Prüfer angle defined in Section 2. Set  $\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa)$ . Then for a.s., the  $t \rightarrow \infty$  limit of  $\tilde{\theta}_t(\kappa)$  exists for any  $\kappa$  [5]. Let  $\tilde{\theta}_\infty(\kappa) := \lim_{t \rightarrow \infty} \tilde{\theta}_t(\kappa)$ .

**Theorem 1.1** ( $\alpha > \frac{1}{2}$ ) *Suppose Assumption A. Then for a.s.,*

$$N_{n_k}(\kappa_1, \kappa_2) - (\lfloor n_k \kappa_2 \rfloor_\pi - \lfloor n_k \kappa_1 \rfloor_\pi) = \left\lfloor \gamma_2 + \tilde{\theta}_\infty(\kappa_2) \right\rfloor_\pi - \left\lfloor \gamma_1 + \tilde{\theta}_\infty(\kappa_1) \right\rfloor_\pi$$

for sufficiently large  $k$ .

(2) **Critical decay** ( $\alpha = \frac{1}{2}$ ) :

**Theorem 1.2** ( $\alpha = \frac{1}{2}$ )

Let  $\{G(\kappa)\}_{\kappa > 0}$ ,  $G$  be mutually independent Gaussian field and a Gaussian such that

$$\text{Cov}(G(\kappa), G(\kappa')) = \frac{1}{2} \delta_{\kappa, \kappa'} \langle [g_\kappa, \bar{g}_\kappa] \rangle, \quad \kappa, \kappa' > 0,$$

$$\begin{aligned}
\text{Cov}(G, G) &= \langle [g, g] \rangle, \\
g_\kappa &= (L + 2i\kappa)^{-1}F, \quad g := L^{-1}(F - \langle F \rangle), \\
[f, g] &:= \nabla f \cdot \nabla g.
\end{aligned}$$

Then as  $n \rightarrow \infty$

$$\begin{aligned}
&\left\{ N_n(\kappa_1, \kappa_2) - \frac{n}{\pi}(\kappa_2 - \kappa_1) - \text{Re} \left( \frac{C_1(\kappa_2)}{2\pi\kappa_2} - \frac{C_1(\kappa_1)}{2\pi\kappa_1} \right) \int_0^n a(s)^2 ds \right\} \frac{1}{\sqrt{\log n}} \\
&\xrightarrow{d} \frac{1}{2\pi\kappa_2} G(\kappa_2) - \frac{1}{2\pi\kappa_1} G(\kappa_1) - \left( \frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1} \right) G
\end{aligned}$$

in the sense of weak convergence as the processes on  $(\kappa_1, \kappa_2) \in (0, \infty)^2$  where  $C_1(\kappa) := -\frac{i}{2\kappa} \langle Fg_\kappa \rangle$  is a deterministic constant.

**Remark 1.1** [1] studied this problem for CMV matrices, but they do not need to subtract the constant term due to the rotational invariance.

### (3) Subcritical-decay ( $\alpha < \frac{1}{2}$ )

#### Theorem 1.3 ( $\alpha < \frac{1}{2}$ )

Set  $D := \min\{d \in \mathbf{N} \mid \frac{1}{2\alpha} < d + 1\}$ . Let  $\{G_t(\kappa)\}_{t \in [0,1], \kappa > 0}$ ,  $\{G_t\}_{t \in [0,1]}$  be the mutually independent Gaussian fields such that

$$\begin{aligned}
\text{Cov}(G_t(\kappa), G_s(\kappa')) &= \frac{1}{2} \delta_{\kappa, \kappa'} \frac{\langle [g_\kappa, \bar{g}_\kappa] \rangle}{1 - 2\alpha} (t \wedge s)^{1-2\alpha} \\
\text{Cov}(G_t, G_s) &= \frac{\langle [g, g] \rangle}{1 - 2\alpha} (t \wedge s)^{1-2\alpha}.
\end{aligned}$$

Then as  $n \rightarrow \infty$

$$\begin{aligned}
&\left\{ N_{nt}(\kappa_1, \kappa_2) - \frac{nt}{\pi}(\kappa_2 - \kappa_1) - \sum_{j=1}^D \text{Re} \left( \frac{C_j(\kappa_2)}{2\pi\kappa_2} - \frac{C_j(\kappa_1)}{2\pi\kappa_1} \right) \int_0^{nt} a(s)^{j+1} ds \right\} \frac{1}{n^{\frac{1}{2}-\alpha}} \\
&\xrightarrow{d} \frac{1}{2\pi\kappa_2} G_t(\kappa_2) - \frac{1}{2\pi\kappa_1} G_t(\kappa_1) - \left( \frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1} \right) G_t
\end{aligned}$$

in the sense of weak convergence as the processes for  $(\kappa_1, \kappa_2, t) \in (0, \infty)^2 \times [0, \infty)$ , where  $C_j(\kappa)$ ,  $j = 1, 2, \dots, D$  are deterministic constants given in (2.6).

**Remark 1.2** For fixed  $\kappa_1, \kappa_2$ , RHS is equal to the superposition of Brownian motions in distribution.

$$\begin{aligned}
&\frac{1}{2\pi\kappa_2} G_t(\kappa_2) - \frac{1}{2\pi\kappa_1} G_t(\kappa_1) - \left( \frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1} \right) G_{0,t} \\
&\stackrel{d}{=} \frac{1}{2\pi\kappa_2} \sqrt{\frac{1}{2} \frac{\langle [g_\kappa, \bar{g}_\kappa] \rangle}{1 - 2\alpha}} B_{t^{1-2\alpha}}^{(2)} - \frac{1}{2\pi\kappa_1} \sqrt{\frac{1}{2} \frac{\langle [g_\kappa, \bar{g}_\kappa] \rangle}{1 - 2\alpha}} B_{t^{1-2\alpha}}^{(1)} \\
&\quad - \left( \frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1} \right) \sqrt{\frac{\langle [g, g] \rangle}{1 - 2\alpha}} B_{t^{1-2\alpha}}^{(0)}.
\end{aligned}$$

**Remark 1.3** If  $a(s)$  satisfies  $a(s) = s^{-\alpha}$ ,  $s \geq R$  for some  $R > 0$ , we have the following asymptotic expansion.

$$N_{nt}(\kappa_1, \kappa_2) \sim \frac{nt}{\pi}(\kappa_2 - \kappa_1) + C_2(nt)^{1-2\alpha} + C_3(nt)^{1-3\alpha} \\ + \dots + C_D(nt)^{1-(D+1)\alpha} + n^{\frac{1}{2}-\alpha}(\text{Gaussian}).$$

**Remark 1.4** Theorems 1.1, 1.2, 1.3 roughly imply that the 2nd term in (1.1) is (1) bounded for  $\alpha > \frac{1}{2}$ , (2)  $O(\log n)$  for  $\alpha = \frac{1}{2}$ , and (3)  $O(n^{1-2\alpha})$  for  $\alpha < \frac{1}{2}$ . That the 2nd term grows bigger as  $\alpha$  becomes smaller reflects the fact that the IDS becomes totally different for  $\alpha = 0$ .

**Remark 1.5** We can study the case where  $a(s)$  decays slower than  $s^{-\alpha}$  for any  $\alpha > 0$ . For instance, if  $a(s) = (\log s + 1)^{-\delta}$  ( $\delta > 0$ ), then

$$N_n(\kappa_1, \kappa_2) = \frac{n}{\pi}(\kappa_2 - \kappa_1) + N_1(\kappa_1, \kappa_2) + N_2(\kappa_1, \kappa_2), \quad n \rightarrow \infty$$

where  $N_1$  has the following asymptotic expansion

$$N_1(\kappa_1, \kappa_2) - \sum_{j=2}^k C_k n (\log n)^{-k\delta} = O\left(n (\log n)^{-(k+1)\delta}\right)$$

with  $C_j$  being deterministic constants.  $N_2$  is a martingale converging to a Gaussian field :

$$\frac{N_2(\kappa_1, \kappa_2)}{n^{1/2}(\log n)^{-\delta}} \xrightarrow{d} G(\kappa_1, \kappa_2).$$

**Remark 1.6** A natural and reasonable extension of the problem discussed in this paper is to consider

$$N_n(f) = \sum_k f\left(\sqrt{E_k(n)}\right)$$

where  $f$  is a sufficiently smooth function compactly supported on  $(0, \infty)$ , and  $\{E_k(n)\}_k$  are the positive eigenvalues of  $H_n$  arranged in the increasing order. We can show that

$$N_n(f) = \sum_j \int_0^\infty f'(\kappa) g_j(\kappa) d\kappa \int_0^n a(s)^j ds + B_n(f) + M_n(f)$$

where  $g_j(\kappa)$  are bounded functions,  $B_n(f)$  is a bounded process, and  $M_n(f)$  is a martingale. However, we are unable to derive the growth order of  $\langle M_n(f) \rangle$  as  $n \rightarrow \infty$  the study of which is postponed to the future work.

**Decaying coupling constant model (DC model) :** Let us consider the following Hamiltonian :

$$H'_n := -\frac{d^2}{dt^2} + \lambda_n F(X_t) \text{ on } L^2[0, n], \quad \lambda_n := n^{-\alpha}, \quad \alpha > 0$$

with Dirichlet boundary condition. Because in 1d, the localization length of  $H = -\Delta + \lambda V$  is typically  $O(\lambda^{-\frac{1}{2}})$ , the property of  $H'_n$  would also change at  $\alpha = \frac{1}{2}$ . In fact, as for the level statistics problem,  $\xi_L$  converges to the (deterministic) clock process for  $\alpha > \frac{1}{2}$ , Sch $_{\tau}$ -process for  $\alpha = \frac{1}{2}$ , and Poisson process for  $\alpha < \frac{1}{2}$  [6, 4]. We can apply the discussion in this paper also to  $H'_n$  and obtain the 2nd term asymptotics of  $N_n(\kappa_1, \kappa_2)$  under the same notation as in Theorems 1.1, 1.2, 1.3. Here the major difference from the decaying potential model  $H$  is that the 2nd term is bounded also for the critical case.

**Theorem 1.4** ( $\alpha > \frac{1}{2}$ ) *Suppose Assumption A with  $\gamma_j \neq 0$ ,  $j = 1, 2$ . Then for a.s.,*

$$N_{n_k}(\kappa_1, \kappa_2) - (\lfloor n_k \kappa_2 \rfloor_{\pi} - \lfloor n_k \kappa_1 \rfloor_{\pi}) = 0$$

for sufficiently large  $k$ .

**Theorem 1.5** ( $\alpha = \frac{1}{2}$ ) *Suppose Assumption A. We then have*

$$N_{n_k}(\kappa_1, \kappa_2) - (\lfloor n_k \kappa_2 \rfloor_{\pi} - \lfloor n_k \kappa_1 \rfloor_{\pi}) \xrightarrow{d} \left[ \gamma_2 + \tilde{\theta}_{\infty}(\kappa_2) \right]_{\pi} - \left[ \gamma_1 + \tilde{\theta}_{\infty}(\kappa_1) \right]_{\pi}$$

as  $k \rightarrow \infty$ .

**Theorem 1.6** ( $\alpha < \frac{1}{2}$ ) *Set  $D := \min\{d \in \mathbf{N} \mid \frac{1}{2\alpha} < d + 1\}$ . Let  $\{G_t(\kappa)\}_{t \in [0, 1], \kappa > 0}$ ,  $\{G_t\}_{t \in [0, 1]}$  be the mutually independent Gaussians such that*

$$\begin{aligned} \text{Cov}(G_t(\kappa), G_s(\kappa')) &= \frac{1}{2} \delta_{\kappa, \kappa'} \langle [g_{\kappa}, \bar{g}_{\kappa}] \rangle (t \wedge s)^{1-2\alpha} \\ \text{Cov}(G_t, G_s) &= \langle [g, g] \rangle (t \wedge s)^{1-2\alpha}. \end{aligned}$$

We then have

$$\begin{aligned} & \left\{ N_{nt}(\kappa_1, \kappa_2) - \frac{nt}{\pi}(\kappa_2 - \kappa_1) - \sum_{j=1}^D \text{Re} \left( \frac{C_j(\kappa_2)}{2\pi\kappa_2} - \frac{C_j(\kappa_1)}{2\pi\kappa_1} \right) (nt)^{1-(j+1)\alpha} \right\} \frac{1}{n^{\frac{1}{2}-\alpha}} \\ & \xrightarrow{d} \frac{1}{2\pi\kappa_2} G_t(\kappa_2) - \frac{1}{2\pi\kappa_1} G_t(\kappa_1) - \left( \frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1} \right) G_t. \end{aligned}$$

The main ingredient of the proof is to express  $N_{nt}(\kappa_1, \kappa_2)$  in terms of the Prüfer angles  $\theta_t(\kappa)$ , as is done in [1]. Then we study the behavior of  $\theta_t(\kappa)$  by the

martingale analysis developed in [5]. The plan of this paper is as follows. In Section 2, we introduce the Prüfer variable and compute the basic integrals which frequently appears in this paper. In Sections 3,4,5, we prove Theorems 1.1, 1.2 and 1.3 respectively.

## 2 Preliminaries

### 2.1 Prüfer coordinate

For  $\kappa > 0$  let  $x_t(\kappa)$  be the solution to the Schrödinger equation  $Hx_t = \kappa^2 x_t$ ,  $x_0(\kappa) = 0$  which we represent in terms of the Prüfer coordinate :

$$\begin{pmatrix} x_t(\kappa) \\ x'_t(\kappa)/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t(\kappa) \\ r_t \cos \theta_t(\kappa) \end{pmatrix}, \quad \theta_0(\kappa) = 0.$$

Let  $\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa)$ . By Sturm's oscillation theorem,

$$N_{nt}(\kappa_1, \kappa_2) - \frac{1}{\pi} n t (\kappa_2 - \kappa_1) = \frac{1}{\pi} \left( \tilde{\theta}_{nt}(\kappa_2) - \tilde{\theta}_{nt}(\kappa_1) \right) \pm 1 \quad (2.1)$$

so that it suffices to study the behavior of  $\tilde{\theta}_t(\kappa)$ . Noting that  $\tilde{\theta}_t(\kappa)$  satisfies the following integral equation,

$$\tilde{\theta}_t(\kappa) = \frac{1}{2\kappa} \operatorname{Re} \int_0^t \left( e^{2i\theta_s(\kappa)} - 1 \right) a(s) F(X_s) ds,$$

we set

$$\begin{aligned} J_t^{(n)}(\kappa) &:= \int_0^{nt} a(s) e^{2i\theta_s(\kappa)} F(X_s) ds \\ J_{0,t}^{(n)} &:= \int_0^{nt} a(s) F(X_s) ds. \end{aligned}$$

We can then decompose

$$\begin{aligned} \tilde{\theta}_{nt}(\kappa_2) - \tilde{\theta}_{nt}(\kappa_1) &= A + B + C \\ A &= \frac{1}{2\kappa_2} \operatorname{Re} J_t^{(n)}(\kappa_2), \quad B = -\frac{1}{2\kappa_1} \operatorname{Re} J_t^{(n)}(\kappa_1), \\ C &= -\left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) \operatorname{Re} J_{0,t}^{(n)}. \end{aligned} \quad (2.2)$$

We shall study the behavior of  $J_t^{(n)}$ ,  $J_{0,t}^{(n)}$ , as  $n \rightarrow \infty$ .

## 2.2 Basic Calculus on Integrals

For  $H \in C^\infty(M)$ , set

$$K_{m,\beta,\kappa,t}^{(n)}(H) := \int_0^{nt} a(s)^m e^{i\beta\theta_s(\kappa)} H(X_s) ds, \quad m \in \mathbf{N}, \beta \in \mathbf{R}.$$

**Lemma 2.1** *If  $\beta \neq 0$ ,*

$$\begin{aligned} K_{m,\beta,\kappa,t}^{(n)}(H) &= K_{m+1,\beta+2,\kappa,t}^{(n)} \left( T_{\beta,\kappa}^+(H) \right) + K_{m+1,\beta-2,\kappa,t}^{(n)} \left( T_{\beta,\kappa}^-(H) \right) \\ &\quad + K_{m+1,\beta,\kappa,t}^{(n)} \left( T_{\beta,\kappa}^0(H) \right) + L_{m,\beta,\kappa,t}^{(n)} + M_{m,\beta,\kappa,t}^{(n)}. \end{aligned} \quad (2.3)$$

where  $T_\beta^\sharp$ ,  $\sharp = \pm, 0$  are the operators acting on  $C^\infty(M)$  defined by

$$\begin{aligned} (T_{\beta,\kappa}^+ H)(x) &= -\frac{i\beta}{2\kappa} \cdot \frac{1}{2} \cdot F(x) \cdot (R_{\beta\kappa} H)(x) \\ (T_{\beta,\kappa}^- H)(x) &= -\frac{i\beta}{2\kappa} \cdot \frac{1}{2} \cdot F(x) \cdot (R_{\beta\kappa} H)(x) \\ (T_{\beta,\kappa}^0 H)(x) &= \frac{i\beta}{2\kappa} \cdot F(x) \cdot (R_{\beta\kappa} H)(x) \\ R_\kappa H &:= (L + i\kappa)^{-1} H. \end{aligned}$$

$L_{m,\beta,\kappa,t}^{(n)}$  is bounded and  $M_{m,\beta,\kappa,t}^{(n)}$  is a martingale such that

$$\begin{aligned} L_{m,\beta,\kappa,t}^{(n)} &= \left[ a(s)^m e^{i\beta\theta_s(\kappa)} R_{\beta\kappa}(H)(X_s) ds \right]_0^{nt} \\ &\quad - \int_0^{nt} (a(s)^m)' e^{i\beta\theta_s(\kappa)} R_{\beta\kappa}(H)(X_s) ds \\ M_{m,\beta,\kappa,t}^{(n)} &= - \int_0^{nt} a(s)^m e^{i\beta\theta_s(\kappa)} (\nabla R_{\beta\kappa} H)(X_s) dX_s \\ \langle M_{m,\beta,\kappa,t}^{(n)}, M_{m,\beta,\kappa,t}^{(n)} \rangle, \langle M_{m,\beta,\kappa,t}^{(n)}, \overline{M_{m,\beta,\kappa,t}^{(n)}} \rangle &= O \left( \int_0^{nt} a(s)^{2m} ds \right), \quad n \rightarrow \infty. \end{aligned}$$

**Remark 2.1** *It is not necessary to distinguish  $T_{\beta,\kappa}^+$  from  $T_{\beta,\kappa}^-$ . We put the plus minus symbol  $\pm$  only to facilitate the computation of some combinatorial quantities in the proof of Proposition 2.3.*

*Proof.* By Ito's formula,

$$e^{i\kappa s} H(X_s) ds = d \left( e^{i\kappa s} R_\kappa(H) \right) - e^{i\kappa s} \nabla R_\kappa(H) dX_s$$

which we substitute into  $K_{m,\beta,\kappa,t}^{(n)}$  and integrate by parts.

$$K_{m,\beta,\kappa,t}^{(n)}(\kappa) = \left[ a(s)^m e^{i\beta\theta_s(\kappa)} R_{\beta\kappa}(H)(X_s) \right]_0^{nt}$$

$$\begin{aligned}
& - \int_0^{nt} (a(s)^m)' e^{i\beta\theta_s(\kappa)} R_{\beta\kappa}(H)(X_s) ds \\
& - \frac{i\beta}{2\kappa} \int_0^{nt} \operatorname{Re} \left( e^{2i\theta_s(\kappa)} - 1 \right) e^{i\beta\theta_s(\kappa)} a(s)^{m+1} F(X_s) R_{\beta\kappa}(H)(X_s) ds \\
& - \int_0^{nt} a(s)^m e^{i\beta\theta_s(\kappa)} \nabla R_{\beta\kappa}(H)(X_s) dX_s \\
& =: K_1(\kappa) + \cdots + K_4(\kappa).
\end{aligned}$$

$K_1, K_2$  are bounded.  $K_3$  gives the first three terms in the RHS of (2.3).  $K_4$  is a martingale and it is easy to check the statement for those.  $\square$

**Lemma 2.2** *If  $\beta = 0$ ,*

$$K_{m,0,t}^{(n)}(H) = \langle H \rangle \int_0^{nt} a(s)^m ds + L_{m,0,t}^{(n)} + M_{m,0,t}^{(n)}$$

where  $L_{m,0,t}^{(n)}$  is bounded, and  $M_{m,0,t}^{(n)}$  is a martingale such that

$$\begin{aligned}
L_{m,0,t}^{(n)} &= [a(s)^m (RH)(X_s)]_0^{nt} - \int_0^{nt} (a(s)^m)' (RH)(X_s) ds \\
\langle M_{m,0,t}^{(n)} \rangle &= O \left( \int_0^{nt} a(s)^{2m} ds \right), \quad n \rightarrow \infty
\end{aligned}$$

where  $(RH)(s) := L^{-1}(H - \langle H \rangle)(s)$ .

*Proof.* By Ito's formula, we have

$$H(X_s) ds = \langle H \rangle ds + d(R(H)(X_s)) - \nabla(R(H))(X_s) dX_s$$

which gives

$$\begin{aligned}
K_{m,0}^{(n)} &= \langle H \rangle \int_0^{nt} a(s)^m ds + [a(s)^m G(X_s)]_0^{nt} \\
&\quad - \int_0^{nt} (a(s)^m)' G(X_s) ds - \int_0^{nt} a(s)^m \nabla G(X_s) dX_s \\
&=: K'_1 + \cdots + K'_4.
\end{aligned}$$

$K'_2, K'_3$  are bounded and  $K'_4$  is a martingale.  $\square$

## 2.3 Expansion of $J$

In this subsection we study the behavior of  $J_t^{(n)}, J_{0,t}^{(n)}$  by using Lemmas 2.1, 2.2.



**Proposition 2.3** *For any  $D \geq 1$  we have*

$$J_t^{(n)}(\kappa) = \sum_{k=1}^D C_k(\kappa) \int_0^{nt} a(s)^{k+1} ds + K_D(\kappa) + L_D(\kappa) + M_D(\kappa) \quad (2.4)$$

where

$$K_D(\kappa) = O\left(\int_0^{nt} a(s)^{D+2} ds\right)$$

and  $L_D$  is bounded. The constants  $C_k(\kappa)$  are given in (2.6) below.  $M_D(\kappa)$  is a martingale such that

$$\begin{aligned} M_D(\kappa) &= M_{1,2,\kappa,t} + M'_D \\ M_{1,2,\kappa,t} &:= - \int_0^{nt} a(s) e^{2i\theta_s(\kappa)} \nabla g_\kappa(X_s) dX_s \\ \langle M_D(\kappa), M_D(\kappa) \rangle &= O\left(\int_0^{nt} a(s)^4 ds\right) \\ \langle M_D(\kappa), \overline{M_D(\kappa)} \rangle &= \langle M_{1,2,\kappa,t}, \overline{M_{1,2,\kappa,t}} \rangle (1 + o(1)) \\ &= \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^{nt} a(s)^2 ds (1 + o(1)) \end{aligned}$$

where we set  $g_\kappa := R_{2\kappa}F$ .

*Proof.*

(1) 1st step : Letting  $m = 1$ ,  $\beta = 2$  in Lemma 2.1,

$$\begin{aligned} J_t^{(n)}(\kappa) &= K_{1,2,\kappa,t}^{(n)}(F) \\ &= K_{2,4,\kappa,t}^{(n)}(T_{2,\kappa}^+(F)) + K_{2,0,\kappa,t}^{(n)}(T_{2,\kappa}^-(F)) + K_{2,2,\kappa,t}^{(n)}(T_{2,\kappa}^0(F)) \\ &\quad + L_{1,2,\kappa,t}^{(n)} + M_{1,2,\kappa,t}^{(n)} \end{aligned} \quad (2.5)$$

We further use Lemma 2.1 to the 1st and 3rd terms in the RHS of (2.5), so that they are  $O\left(\int_0^{nt} a(s)^3 ds\right)$ . For the 2nd term  $K_{2,0,\kappa,t}^{(n)}$ , we use Lemma 2.2.

$$\begin{aligned} K_{2,0,\kappa,t}^{(n)}(T_{2,\kappa}^-(F)) &= \langle T_{2,\kappa}^-(F) \rangle \int_0^{nt} a(s)^2 ds + L_{2,0,t}^{(n)} + M_{2,0,t}^{(n)}, \\ \langle M_{2,0,t}^{(n)}, M_{2,0,t}^{(n)} \rangle, \langle M_{2,0,t}^{(n)}, \overline{M_{2,0,t}^{(n)}} \rangle &= O\left(\int_0^{nt} a(s)^4 ds\right). \end{aligned}$$

For the 5th martingale term  $M_{1,2,\kappa,t}^{(n)}$  in the RHS of (2.5), we estimate its quadratic variation by Lemmas 2.1, 2.2.

$$M_{1,2,\kappa,t}^{(n)} = - \int_0^{nt} a(s) e^{2i\theta_s(\kappa)} (\nabla g_\kappa)(X_s) dX_s$$

$$\begin{aligned}
\langle M_{1,2,\kappa,t}^{(n)}, M_{1,2,\kappa,t}^{(n)} \rangle &= K_{2,4,\kappa,t}(\varphi_\kappa), \quad \varphi_\kappa := [g_\kappa, \bar{g}_\kappa] \\
&= K_{3,6,\kappa,t}(T_{\beta,\kappa}^+(\varphi_\kappa)) + K_{3,2,\kappa,t}(T_{\beta,\kappa}^-(\varphi_\kappa)) + K_{3,4,\kappa,t}(T_{\beta,\kappa}(\varphi_\kappa)) \\
&\quad + L_{2,4,\kappa,t} + M_{2,4,\kappa,t}.
\end{aligned}$$

For the first three terms of RHS we use Lemma 2.1 again. Moreover we have  $\langle M_{2,4,\kappa,t}, M_{2,4,\kappa,t} \rangle = O\left(\int_0^{nt} a(s)^4 ds\right) = O(n^{1-4\alpha})$ . Since  $1 - 3\alpha > \frac{1-4\alpha}{2}$  if and only if  $\alpha < \frac{1}{2}$ , it is lower order or bounded. Therefore

$$\begin{aligned}
\langle M_{1,2,\kappa,t}^{(n)}, M_{1,2,\kappa,t}^{(n)} \rangle &= O\left(\int_0^{nt} a(s)^4 ds\right) \\
\langle M_{1,2,\kappa,t}^{(n)}, \bar{M}_{1,2,\kappa,t}^{(n)} \rangle &= \int_0^{nt} a(s)^2 [g_\kappa, \bar{g}_\kappa](X_s) ds \\
&= \langle [g_\kappa, \bar{g}_\kappa] \rangle \int_0^{nt} a(s)^2 ds + (\text{bounded}) + M'_{1,2,\kappa,t} \\
\langle M'_{1,2,\kappa,t}, M'_{1,2,\kappa,t} \rangle &= O\left(\int_0^{nt} a(s)^4 ds\right)
\end{aligned}$$

and (2.4) is proved for  $D = 1$ .

(2)  $(k+1)$ -th step : we iterate this process. After the  $k$ -th step, we have a sum of  $\sum_{j=1}^k C_j(\kappa) \int_0^{nt} a(s)^{j+1} ds$ ,  $K_{k+1,\beta,\kappa,t}(H)$  ( $\beta \neq 0$ ), bounded term, and a sum of martingales. So in the  $(k+1)$ -th step, we apply Lemma 2.1 to  $K_{k+1,\beta,\kappa,t}(H)$  ( $\beta \neq 0$ ) to have  $K_{k+2,\beta',\kappa,t}(H)$  ( $\beta' \neq 0$ ),  $K_{k+2,0,\kappa,t}(H)$ , bounded term and a sum of martingales. We further apply Lemma 2.2 to  $K_{k+2,0,\kappa,t}(H)$  to have a deterministic term proportional to  $\int_0^{nt} a(s)^{k+2} ds$  and a sum of bounded terms and martingales. We note that the martingales which emerge in each steps have quadratic variation with order at most  $O\left(\int_0^{nt} a(s)^4 ds\right)$ , except  $M_{1,2,\kappa,t}$ . Letting  $M_K$  be the sum of all martingales appeared up to the  $D$ -th step,  $M_K$  satisfies the statement in Proposition 2.3. To compute the coefficients proportional to  $\int_0^{nt} a(s)^{k+1} ds$ , we consider a set of indices :

$$S_k := \left\{ ((\epsilon_1, \dots, \epsilon_{k-1}), (\beta_1, \dots, \beta_{k-1})) \left| \begin{aligned} &\epsilon_i = 0, \pm 1, \\ &\beta_i \in 2\mathbf{N}, \beta_{i+1} = \beta_i + 2\epsilon_i, \beta_1 = 2 \\ &\sum_{i=1}^j \epsilon_i \geq 0, 1 \leq j \leq k-2, \sum_{i=1}^{k-1} \epsilon_i = 0 \end{aligned} \right. \right\}$$

then the desired coefficients is given by

$$C_k(\kappa) = \begin{cases} \langle T_{2,\kappa}^{-1} F \rangle & (k=1) \\ \sum_{((\epsilon_i), (\beta_i)) \in S_k} \langle T_{2,\kappa}^{-1} T_{\beta_{k-1},\kappa}^{\epsilon_{k-1}} \cdots T_{\beta_1,\kappa}^{\epsilon_1} F \rangle & (k \geq 2) \end{cases} \quad (2.6)$$

For instance, omitting the  $\kappa$ -dependence, we have

$$\begin{aligned} C_2 &= \langle T_2^{-1} T_2^0 F \rangle \\ C_3 &= \langle T_2^- T_4^- T_2^+ F + T_2^- T_2^0 T_2^0 F \rangle. \end{aligned}$$

Proof of Proposition 2.3 is now complete.  $\square$

Lemma 2.2 yields the following decomposition of  $J_{0,t}^{(n)}$ .

**Proposition 2.4**

$$\begin{aligned} J_{0,t}^{(n)} &= L_{0,nt} + M_{0,nt} \\ L_{0,nt} &= -a(0)g(X_0) - \int_0^{nt} a'(s)g(X_s)ds \\ M_{0,nt} &= - \int_0^{nt} a(s)\nabla g(X_s)dX_s \end{aligned} \quad (2.7)$$

where  $L_{0,nt}$  is bounded,  $M_{0,nt}$  is a martingale such that

$$\langle M_0, M_0 \rangle = \langle [g, g] \rangle \int_0^{nt} a(s)^2 ds (1 + o(1)).$$

### 3 Proof for supercritical case

First of all, we notice that the argument of the proof of Proposition 7.1 in [3] shows that the distribution of  $\tilde{\theta}_\infty(\kappa)$  is continuous for  $\alpha > \frac{1}{2}$  (also for DC model with  $\alpha \geq \frac{1}{2}$ ). In fact, we can show that  $\lim_{m \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}[e^{im\tilde{\theta}_t(\kappa)}] = 0$ . Thus  $\left\{ \gamma_j + \tilde{\theta}_\infty(\kappa_j) \right\}_\pi \neq 0$ , a.s. so that

$$\theta_{n_k}(\kappa_j) = \lfloor \kappa_j n_k \rfloor_\pi \pi + \lfloor \gamma_j + \tilde{\theta}_\infty(\kappa_j) \rfloor_\pi \pi + \left\{ \gamma_j + \tilde{\theta}_\infty(\kappa_j) \right\}_\pi + o(1), \quad j = 1, 2, \quad a.s.$$

Theorem 1.1 now follows from Sturm's oscillation theorem.  $\square$

### 4 Proof for critical case

Using Proposition 2.3 with  $D = 1$  and substituting it into (2.2) yields

$$\tilde{\theta}_n(\kappa_2) - \tilde{\theta}_n(\kappa_1)$$

$$\begin{aligned}
&= \left( \frac{ReC_1(\kappa_2)}{2\kappa_2} - \frac{ReC_1(\kappa_1)}{2\kappa_1} \right) \int_0^n a(s)^2 ds \\
&\quad + \frac{1}{2\kappa_2} Re(K_1(\kappa_2) + L_1(\kappa_2)) - \frac{1}{2\kappa_1} Re(K_1(\kappa_1) + L_1(\kappa_1)) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) L_0 \\
&\quad + \frac{1}{2\kappa_2} ReM_1(\kappa_2) - \frac{1}{2\kappa_1} ReM_1(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) M_0.
\end{aligned}$$

By Lemmas 2.1, 2.2,

$$\begin{aligned}
K_1(\kappa) &= O\left(\int_0^n a(s)^4 ds\right) \\
\langle ReM_1(\kappa), ReM_1(\kappa') \rangle &= \frac{1}{2} \langle [g_\kappa, \bar{g}_\kappa] \rangle \log n (\delta_{\kappa, \kappa'} + o(1)) \\
\langle ReM_1, M_0 \rangle &= o(\log n) \\
\langle M_0, M_0 \rangle &= \langle [g, g] \rangle \log n (1 + o(1)).
\end{aligned}$$

Let

$$M(\kappa_1, \kappa_2) := \frac{1}{2\kappa_2} ReM_1(\kappa_2) - \frac{1}{2\kappa_1} ReM_1(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) M_0$$

be its martingale part. Let  $G(\kappa)$ ,  $G$  be independent Gaussians satisfying the covariance condition stated in Theorem 1.2. Then by the martingale central limit theorem,

$$\frac{M(\kappa_1, \kappa_2)}{\sqrt{\log n}} \xrightarrow{d} \frac{1}{2\kappa_2} G(\kappa_2) - \frac{1}{2\kappa_1} G(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) G$$

which leads us to the completion of proof :

$$\begin{aligned}
&\left\{ \tilde{\theta}_n(\kappa_2) - \tilde{\theta}_n(\kappa_1) - Re \left( \frac{C_1(\kappa_2)}{2\kappa_2} - \frac{C_1(\kappa_1)}{2\kappa_1} \right) \int_0^n a(s)^2 ds \right\} \frac{1}{\sqrt{\log n}} \\
&= \left\{ \frac{1}{2\kappa_2} Re(K_1(\kappa_2) + L_1(\kappa_2)) - \frac{1}{2\kappa_1} Re(K_1(\kappa_1) + L_1(\kappa_1)) \right. \\
&\quad \left. - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) L_0 \right\} \frac{1}{\sqrt{\log n}} + \frac{M(\kappa_1, \kappa_2)}{\sqrt{\log n}} \\
&\xrightarrow{d} \frac{1}{2\kappa_2} G(\kappa_2) - \frac{1}{2\kappa_1} G(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) G.
\end{aligned}$$

□

## 5 Proof for subcritical case

Let  $D := \min\{d \in \mathbf{N} \mid \frac{1}{2\alpha} < d + 1\}$ . Substituting (2.4), (2.7) into (2.2), yields

$$\tilde{\theta}_{nt}(\kappa_2) - \tilde{\theta}_{nt}(\kappa_1)$$

$$\begin{aligned}
&= \sum_{j=1}^D \left( \frac{ReC_j(\kappa_2)}{2\kappa_2} - \frac{ReC_j(\kappa_1)}{2\kappa_1} \right) \int_0^{nt} a(s)^{j+1} ds \\
&\quad + \frac{1}{2\kappa_2} Re(K_D(\kappa_2) + L_D(\kappa_2)) - \frac{1}{2\kappa_1} Re(K_D(\kappa_1) + L_D(\kappa_1)) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) L_0 \\
&\quad + \frac{1}{2\kappa_2} ReM_D(\kappa_2) - \frac{1}{2\kappa_1} ReM_D(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) M_0.
\end{aligned}$$

Let

$$M(\kappa_1, \kappa_2) := \frac{1}{2\kappa_2} ReM_D(\kappa_2) - \frac{1}{2\kappa_1} ReM_D(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) M_0$$

be the martingale part. We estimate the quadratic variations of  $M_D(\kappa_2)$ ,  $M_D(\kappa_1)$ ,  $M_0$  by using Lemmas 2.1, 2.2.

$$\begin{aligned}
\langle ReM_D(\kappa), ReM_D(\kappa') \rangle &= \frac{1}{2} \langle [g_\kappa, \bar{g}_\kappa] \rangle \frac{n^{1-2\alpha}}{1-2\alpha} t^{1-2\alpha} (\delta_{\kappa, \kappa'} + o(1)) \\
\langle ReM_D, M_0 \rangle &= o(n^{1-2\alpha}) \\
\langle M_0, M_0 \rangle &= \langle [g, g] \rangle \frac{n^{1-2\alpha}}{1-2\alpha} t^{1-2\alpha} (1 + o(1)).
\end{aligned}$$

Thus letting  $\{G_t(\kappa)\}$ ,  $\{G_t\}$  be the Gaussians defined in the statement of Theorem 1.3, we have

$$\frac{M(\kappa_1, \kappa_2)}{n^{\frac{1}{2}-\alpha}} \xrightarrow{d} \frac{1}{2\kappa_2} G_t(\kappa_2) - \frac{1}{2\kappa_1} G_t(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) G_t.$$

On the other hand, since

$$K_D = O \left( \int_0^{nt} a(s)^{D+2} ds \right) = O \left( n^{1-(D+2)\alpha} \right)$$

$K_D$  is lower order compared to the martingale terms. Therefore

$$\begin{aligned}
&\left\{ \tilde{\theta}_{nt}(\kappa_2) - \tilde{\theta}_{nt}(\kappa_1) - \sum_{j=1}^D Re \left( \frac{C_j(\kappa_2)}{2\kappa_2} - \frac{C_j(\kappa_1)}{2\kappa_1} \right) \int_0^{nt} a(s)^{j+1} ds \right\} \frac{1}{n^{\frac{1}{2}-\alpha}} \\
&= \left\{ \frac{1}{2\kappa_2} Re(K_D(\kappa_2) + L_D(\kappa_2)) - \frac{1}{2\kappa_1} Re(K_D(\kappa_1) + L_D(\kappa_1)) \right. \\
&\quad \left. - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) L_0 \right\} \frac{1}{n^{\frac{1}{2}-\alpha}} + \frac{M(\kappa_1, \kappa_2)}{n^{\frac{1}{2}-\alpha}} \\
&\xrightarrow{d} \frac{1}{2\kappa_2} G_t(\kappa_2) - \frac{1}{2\kappa_1} G_t(\kappa_1) - \left( \frac{1}{2\kappa_2} - \frac{1}{2\kappa_1} \right) G_{0,t}
\end{aligned}$$

completing the proof of Theorem 1.3.  $\square$

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